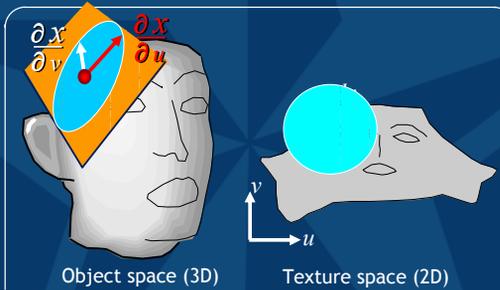


Parameterization

Anisotropy - partial derivatives



Object space (3D)

Texture space (2D)

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Parameterization

Anisotropy - 1st fundamental form



$$G = \begin{bmatrix} \frac{\partial x^2}{\partial u} & \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} & \frac{\partial x^2}{\partial v} \end{bmatrix}$$

$$\|X(W)\|^2 = W^t \cdot G \cdot W$$

$$a = \sqrt{\lambda_1} ; b = \sqrt{\lambda_2} \quad (\text{eigen values of } G)$$

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Parameterization

Anisotropy - Jacobian



$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = U \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} V^t$$

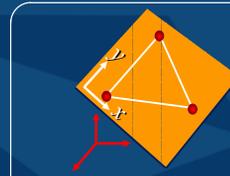
Singular values decomposition (SVD) of J

$$\text{Rem: } G = J^t \cdot J \iff a = \sqrt{\lambda_1} ; b = \sqrt{\lambda_2}$$

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Parameterization

Piecewise Linear Parameterization



From (x, y, z) to (u, v)

$$u(x, y) = a_1 \cdot x + b_1 \cdot y + c_1$$

$$v(x, y) = a_2 \cdot x + b_2 \cdot y + c_2$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

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Parameterization

G coefficients - Previous Work



- [Maillot 93]
 - $\|G - I\|^2$
- [Hormann 00] MIPS :
 - $\|J\|_F \|J^{-1}\|_F = a/b + b/a = \text{trace}(G) / \det(J)$
- [Sander 01] Stretch minimization :
 - $L^2 = \sqrt{a^2 + b^2} / 2 ; L^\infty = \max(a, b)$
- Conformal Maps, Dirichlet Energy
 - [Pinkall93], [Eck95], [Haker00], [Desbrun01], [Sheffer01]

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Parameterization

other methods

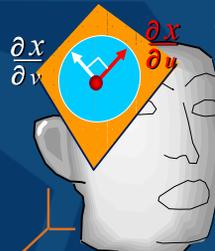


- Barycentric Maps [Floater 95], [Levy 98]
- Spectral methods, MDS [Zigelman]
- Gradient Regularization [Levy 01]
- Compatible triangulations [Gotsmann]

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Parameterization

Notion of conformality



Conformal = locally isotropic
 $a = b$
 Laplace – Beltrami Δ

$$\frac{\partial x}{\partial v} \wedge N = \frac{\partial x}{\partial u}$$

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Parameterization

Notion of conformality



Cauchy-Riemann: $\begin{cases} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \end{cases}$

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Parameterization

Link with Dirichlet Energy

$$E_C(u) + A_u(T) = E_D(u)$$

where:

$$E_D(u) = \frac{1}{2} \int |\nabla u|^2 \quad \text{Dirichlet Energy}$$

$$A_u(T) = \int \det(J_u) \quad \text{Area of } T$$

$$E_C(u) = \frac{1}{2} \int \|D^{90}(\partial u) - \partial v\|^2 \quad \text{Conformal Energy}$$

[Douglas31] [Rado30] [Courant50] [Brakke90]

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Parameterization

The cotangent coefficients [Pinkall93]



$$E_D(u) = \frac{1}{2} \int |\nabla u|^2 = \sum \cot(\alpha_i) \cdot a_i^2$$

[Pinkall93], [Eck95], [Haker00], [Desbrun01]

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LSCM

Strategy

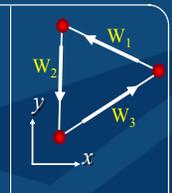
- **Sum of Squares** \Leftrightarrow **Gramm matrices**
- **Topo. Disc** \Leftrightarrow **Euler operators**
- **Similarity Invariance** \Leftrightarrow \mathbb{C}

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LSCM

Cauchy-Riemann in a Δ : back to the roots (of -1)

$$\frac{\partial v}{\partial x} + i \frac{\partial v}{\partial y} = i \left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right)$$

$$E_C(T) = \frac{1}{2A} \left| \begin{bmatrix} W_1 & W_2 & W_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \right|^2$$


$$W_k = (x_{k+2} - x_{k+1}) + i(y_{k+2} - y_{k+1}) \quad ; \quad U_k = u_k + i v_k$$

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LSCM



Cauchy-Riemann in a Δ : back to the roots (of -1)

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \xrightarrow{f} [W_1 \ W_2 \ W_3] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$\text{Ker}(f) = \text{Span}\left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

$R \cdot e^{i\theta} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + T \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

LSCM



Matrix Form of the Criterion

$$E_C([U_1, \dots, U_n]^t) = \sum_T E_C(T) = \bar{U}^t \cdot \mathbf{c} \cdot U$$

where $\mathbf{c} = \bar{M}^t M$

and $m_{i,j} = \begin{cases} W_{j,T_i} / \sqrt{2 \cdot A(T_i)} & \text{if } j \text{ in } T_i \\ 0 & \text{otherwise} \end{cases}$

LSCM



Removing degrees of freedom

$$C = \begin{bmatrix} M_f & M_p \end{bmatrix} \begin{bmatrix} uf_1 \\ \vdots \\ uf_{nf} \\ up_1 \\ \vdots \\ up_{np} \end{bmatrix}_H^2$$

$U = [U_f; U_p]^t$ $M = [M_f; M_p]$

$E_C(U_p) = \|M_f \cdot U_f + M_p \cdot U_p\|_H^2$

LSCM



Back to Reality

$$E_C(\mathbf{x}) = \| \mathbf{A} \cdot \mathbf{x} - \mathbf{b} \|^2 \quad a+ib \equiv \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \text{Re}(M_f) & -\text{Im}(M_f) \\ \text{Im}(M_f) & \text{Re}(M_f) \end{bmatrix}$$

$$\mathbf{b} = - \begin{bmatrix} \text{Re}(M_p) & -\text{Im}(M_p) \\ \text{Im}(M_p) & \text{Re}(M_p) \end{bmatrix} \begin{bmatrix} \text{Re}(U_p) \\ \text{Im}(U_p) \end{bmatrix}$$

LSCM



Critical point of E_c

$$E_C(\mathbf{x}) = \| \mathbf{A} \cdot \mathbf{x} - \mathbf{b} \|^2$$

if $A^t \cdot A$ is non-singular, the critical point \mathbf{x}^* is given by:

$$\mathbf{x}^* = (A^t \cdot A)^{-1} \cdot A^t \cdot \mathbf{b}$$

Rem: $A^t \cdot A$ is the Gramm matrix of the columns of A

$\{A \text{ is of maximum rank}\} \Rightarrow \{A^t \cdot A \text{ is non-singular}\}$

LSCM



Euler Operators

- Two operators, *glue* and *join*
- Preserve the maximal rank property of A

- Vertices -> columns of A
- Triangles -> rows of A

LSCM

The join operator



$$M_f^{(i)} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ W_{1,T} & W_{2,T} & W_{3,T} & 0..0 \end{pmatrix}$$

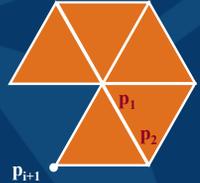


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The glue operator



$$M_f^{(i)} = \begin{pmatrix} \dots & \dots & \dots & \dots & 0 \\ W_{1,T} & W_{2,T} & 0..0 & W_{i+1,T} \\ \lambda_1 & \lambda_2 & \dots & \lambda_{i+1} \end{pmatrix}$$



LSCM - other properties

Triangle Flips - erratum



- For internal vertices, LSCM = cotangent formula
- [Desbrun02] = LSCM
 ⇒ Triangle flips may occur.
 Counter-example (see web-page)
 In practice: never observed

How to fix this:

- Constrained optimization as in [Sheffer]
- Rivara's subdivision as in [Desbrun]

LSCM

Other properties



- Natural border extrapolation
- Taken into account by quadratic form
- Independance to a similarity applied to the pinned vertices
- Independance to resolution

LSCM

Example: a huge chart (72500 trgl's)

